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JEE Class Companion Mathematics

For JEE Main and Advanced

Module - 1 + 2

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Contents

JEE SYLLABUS

- **BASIC MATHEMATICS & LOGARITHAM**

  Logarithms and their properties

- **QUADRATIC EQUATION**

  Quadratic equations with real coefficients, relations between roots and coefficients, formation of quadratic equations with given roots, symmetric functions of roots

- **SEQUENCE & SERIES**

  Arithmetic, geometric and harmonic progressions, arithmetic, geometric and harmonic means, sums of finite arithmetic and geometric progressions, infinite geometric series, sums of squares and cubes of the first n natural numbers.

- **TRIGONOMETRIC RATIOS & IDENTITIES (PHASE - I)**

  Trigonometric functions, their periodicity and graphs, addition and subtraction formulae, formulae involving multiple and sub-multiple angles

- **TRIGONOMETRIC EQUATION (PHASE - II)**

  General solution of trigonometric equations.

- **SOLUTION OF TRIANGLE (PH-III)**

  Relations between sides and angles of a triangle, sine rule, cosine rule, half-angle formula and

  the area of a triangle.

- **COMPLEX NUMBER**

  The Real number system, Imaginary number, Complex number, Modulus of a complex number, Amplitude of a complex number, Square root of a complex number, Triangle inequalities, Miscellaneous results

CHAPTER 1

Basic Mathematic & Logarithm

SECTION - A : BASIC MATHS

NUMBER SYSTEM

Natural Numbers

The counting numbers 1, 2, 3, 4 .... are called Natural Numbers. The set of natural numbers is denoted by N. Thus, N = {1, 2, 3, 4,......}. N is also denoted by I^+ or Z^+

Whole Numbers

Natural numbers including zero are called whole numbers. The set of whole numbers, is denoted by W. Thus W = {0, 1, 2,......}. W is also called as set of non-negative integers.

Integers

The numbers..– 3, – 2, –1, 0, 1, 2, ... are called integers and the set is denoted by I or Z.

Thus I (or Z) = {...–3, –2, –1, 0, 1, 2, 3.........}

1. Set of positive integers, denoted by I^+ and consists of {1, 2, 3, ...}
2. Set of negative integers, denoted by I^- and consists of {..........., –3, –2, –1}
3. Set of non-negative integers {0, 1, 2, 3,...........}
4. Set of non-positive integers {...., –3, –2, –1, 0}

Even Integers

Integers which are divisible by 2 are called even integers. e.g. 0, ± 2, ± 4,.....

Odd Integers

Integers which are not divisible by 2 are called as odd integers. e.g. ± 1, ± 3, ...

Prime Number

Let 'p' be a natural number, 'p' is said to be prime if it has exactly two distinct factors, namely 1 and itself.
e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31,......

Remarks

1. '1' is neither prime nor composite.
2. '2' is the only even prime number.

Composite Number

Let 'a' be a natural number, 'a' is said to be composite if, it has atleast three distinct factors.

Co-prime Numbers

Two natural numbers (not necessarily prime) are coprime, if their H.C.F.(Highest common factor) is one.e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) etc.

These numbers are also called as **relatively prime** numbers.

Remarks

1. Number which are not prime are composite numbers (except 1)
2. '4' is the smallest composite number.
3. Two distinct prime numbers are always co-prime but converse need not be true.
4. Consecutive numbers are always co-prime numbers.

Twin Prime Numbers

If the difference between two prime numbers is two, then the numbers are called as twin prime numbers.
eg. {3, 5}, {5, 7}, {11, 13}, {17, 19}, {29, 31}

Rational Numbers

All the numbers those can be represented in the form p/q, where p and q are integers and q ≠ 0, are called rational numbers and their set is denoted by Q.

Thus Q = $\{\frac{p}{q} : p, q \in I \text{ and } q \neq 0\}$. It may be noted that every integer is a rational numbers.If not integer then either finite or recurring.

Irrational Numbers

There are real numbers which cannot be expressed in p/q form. These numbers are Called irrational numbers and their set is denoted by Q^c or Q'.

(i.e. complementary set of Q) e.g. $\sqrt{2}$, $1+\sqrt{3}$, e, π etc. Irrational numbers can not be expressed as recurring decimals.

Remark :

1. e. ≈ 2.71 is called Napier's constant and π ≈ 3.14.

SURDS

If a is not a perfect nth power, then $\sqrt[n]{a}$ is called a surd of the nth order.

In an expression of the form $\frac{a}{\sqrt{b}+\sqrt{c}}$, the denominator can be rationalized by multiplying numerator and the denominator by $\sqrt{b}-\sqrt{c}$ which is called the conjugate of $\sqrt{b}+\sqrt{c}$. If $x+\sqrt{y} = a+\sqrt{b}$ where x, y, a, b are rationals, then x = a and y = b.

SOLVED EXAMPLE

EXAMPLE 1

Prove that $\log_3 5$ is irrational.

SOLUTION

Let $\log_3 5$ is rational.

$\therefore \log_3 5 = \frac{p}{q}$; where p and q are co-prime numbers

$\Rightarrow 3^{p/q} = 5 \Rightarrow 3^p = 5^q$. which is not possible, hence our assumption is wrong and $\log_3 5$ is irrational.

EXAMPLE 2

Simply (make the denominator rational) $\frac{12}{3+\sqrt{5}-2\sqrt{2}}$

SOLUTION

The expression = $\frac{12(3+\sqrt{5}+2\sqrt{2})}{(3+\sqrt{5})^2-(2\sqrt{2})^2} = \frac{12(3+\sqrt{5}+2\sqrt{2})}{6+6\sqrt{5}}$

$= \frac{2(3+\sqrt{5}+2\sqrt{2})(\sqrt{5}-1)}{(\sqrt{5}+1)\times(\sqrt{5}-1)} = \frac{2(2+2\sqrt{5}+2\sqrt{10}-2\sqrt{2})}{4}$

$= 1+\sqrt{5}+\sqrt{10}-\sqrt{2}$

EXAMPLE 3

Find the factor which will rationalize $\sqrt{3}+\sqrt[3]{5}$

SOLUTION

Let $x = 3^{1/2}$ and $y = 5^{1/3}$. The L.C.M. of the denominators of the indices 2 and 3 is 6. Hence x^6 and y^6 are rational. Now $x^6 + y^6 = (x + y)(x^5 - x^4 y + x^3 y^2 - x^2y^3 + xy^4 - y^5)$

Hence the rationalizing factor required = $x^5 - x^4 y + x^3 y^2 - x^2 y^3 + xy^4 - y^5$ where $x = 3^{1/2}$ and $y = 5^{1/3}$.

EXAMPLE 4

Find the square root of $7 + 2\sqrt{10}$

SOLUTION

Let $\sqrt{7+2\sqrt{10}} = \sqrt{x}+\sqrt{y}$. Squaring, $x + y + 2\sqrt{xy} = 7+2\sqrt{10}$

Hence x + y = 7 and xy = 10.These two relation give x = 5, y = 2. Hence $\sqrt{7+2\sqrt{10}} = \sqrt{5}+\sqrt{2}$

Remark :

1. $\sqrt{\ }$ symbol stands for the positive square root only.

EXAMPLE 5

Prove that $\sqrt[3]{2}$ cannot be represented in the form $p+\sqrt{q}$, where p and q are rational (q > 0 and is not a perfect square).

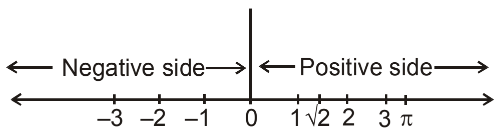
SOLUTION

Put $\sqrt[3]{2} = p+\sqrt{q}$. Hence $2 = p^3 + 3pq + (3p^2 + q)\sqrt{q}$,
Since q is not a perfect square, it must be $3p^2 + q = 0$, which is impossible.

Real Numbers

The complete set of rational and irrational numbers is the set of real numbers and is denoted by R. Thus $R = Q \cup Q^c$. Real numbers can be represented as points of a line. This line is called as real line or number line

All the real numbers follow the order property i.e. if there are two distinct real numbers a and b then either a < b or a > b



Remarks

1. Integers are rational numbers, but converse need not be true.
2. Negative of an irrational number is an irrational number.
3. Sum of a rational number and an irrational number is always an irrational number e.g. $2+\sqrt{3}$
4. The product of a non zero rational number & an irr. number will always be an irrational number.
5. If $a \in Q$ and $b \notin Q$, then ab = rational number, only if a = 0.
6. Sum, difference, product and quotient of two irrational numbers need not be an irrational number (it may be a rational number also).

Complex Number

A number of the form a + ib is called complex number, where a, b ∈ R and $i = \sqrt{-1}$. Complex number is usually denoted by C.

Remark

1. It may be noted that $N \subset W \subset I \subset Q \subset R \subset C$.

SOLVED EXAMPLE

EXAMPLE 6

Every number is one of the forms 5n, 5n ± 1, 5n ± 2.

SOLUTION

For if any number is divided by 5, the remainder is one of the numbers 0, 1, 2, 5 – 2, 5 – 1.

EXAMPLE 7

Every square number is one of the forms 5n, 5n ± 1.

SOLUTION

The square of every number is one of the forms $(5m)^2$, $(5m \pm 1)^2$, $(5m \pm 2)^2$. If those are divided by 5, the remainders are 0, 1, 4; and, since 4 = 5 – 1, the forms are 5n, 5n + 1, and 5n – 1.

EXAMPLE 8

Show that the number of primes in N is infinite.

SOLUTION

Suppose the number of primes in N is finite. Let $\{p_1, p_2, ..., p_n\}$ be the set of primes in N such that $p_1 < p_2 < .... < p_n$. Consider $n = 1 + p_1p_2......p_n$. clearly n is not divisible by any one of $p_1, p_2, ..., p_n$. Hence n itself is a prime and n has a prime divisor other than $p_1, p_2....p_n$. This contradicts that the set of primes is $\{p_1, p_2....,p_n\}$. Therefore the number of primes in N is infinite.

EXAMPLE 9

If x and y are prime numbers which satisfy $x^2 – 2y^2 = 1$, solve for x and y

SOLUTION

$x^2 – 2y^2 = 1$ gives $x^2 = 2y^2 + 1$ and hence x must be an odd number.
If x = 2n + 1, then $x^2 = (2n+1)^2 = 4n^2 + 4n + 1 = 2y^2+1$
Therefore $y^2 = 2n(n + 1)$. This means that y^2 is even and hence y is an even integer. Now, y is also a prime implies that y = 2. This gives x = 3. Thus the only solution is x = 3, y = 2.

DIVISIBILITY TEST

1. A number will be divisible by 2 iff the digit at the unit place is divisible by 2.
2. A number will be divisible by 3 iff the sum of its digits of the number is divisible by 3.
3. A number will be divisible by 4 iff last two digits of the number together are divisible by 4.
4. A number will be divisible by 5 iff digit at the unit place is either 0 or 5.
5. A number will be divisible by 6 iff the digit at the unit place of the number is divisible by 2 & sum of all digits of the number is divisible by 3.
6. A number will be divisible by 8 iff the last 3 digits, all together, is divisible by 8.
7. A number will be divisible by 9 iff sum of all it's digits is divisible by 9.
8. A number will be divisible by 10 iff it's last digit is 0.
9. A number will be divisible by 11 iff the difference between the sum of the digits at even places and sum of the digits at odd places is a multiple of 11.
   Example. 1298, 1221, 123321, 12344321, 1234554321, 123456654321, 795432

SOLVED EXAMPLE

EXAMPLE 10

Prove that :

1. The sum $\overline{ab} + \overline{ba}$ is multiple of 11;
2. A three-digit number written by one and the same digit is entirely divisible by 37.

SOLUTION

1. $\overline{ab} + \overline{ba} = (10a + b) + (10b + a) = 11(a + b);$
2. $\overline{aaa} = 100a + 10a + a = 111a = 37.3a.$

EXAMPLE 11

Prove that the difference $10^{25} – 7$ is divisible by 3.

SOLUTION

Write the given difference in the form $10^{25} – 7 = (10^{25} – 1) – 6$.
The number $10^{25} – 1 = \underbrace{99..9}_{25\text{ digits}}$ is divisible by 3(and 9). Since the numbers $(10^{25} – 1)$ and 6 are divisible by 3, the number $10^{25} – 7$, being their difference, is also divisible by 3 without a remainder.

EXAMPLE 12

If the number A 3 6 4 0 5 4 8 9 8 1 2 7 0 6 4 4 B is divisible by 99 then the ordered pair of digits (A, B) is

SOLUTION

$S_O = A + 37$; $S_E = B + 34$ $\Rightarrow$ A – B + 3 = 0 or 11
and A + B + 71 is a multiple of 9
$\Rightarrow$ A – B = – 3 or 8 and A + B = 1 or 10 **Ans.** : (9, 1)

EXAMPLE 13

Consider a number N = 2 1 P 5 3 Q 4. Find the number of ordered pairs (P, Q) so that the number 'N is divisible by 44, is

SOLUTION

$S_O = P + 9$, $S_E = Q + 6$ $\Rightarrow$ $S_O - S_E = P - Q + 3$
'N' is divisible is 11 if P – Q +3 = 0, 11

P – Q = –3 ....(i)
or P – Q = 8 ....(ii)

N is divisible by 4 if Q = 0, 2, 4, 6, 8
From Equation (i)
Q = 0 P = –3 (not possible)
Q = 2 P = –1 (not possible)
Q = 4 P = 1 Q = 6 P = 3 Q = 8 P = 5
$\therefore$ number of ordered pairs is 3
From equation (ii)
Q = 0 P = 8 Q = 2 P = 10 (not possible) similarly $Q \neq 4, 6, 8$
$\therefore$ No. of ordered pairs is 1
$\therefore$ total number of ordered pairs, so that number 'N' is divisible by 44, is 4

EXAMPLE 14

Prove that the square of any prime number $p \geq 5$, when divided by 12, gives 1 as remainder.

SOLUTION

When divided by 6, a natural number can give as a remainder only the numbers 0, 1, 2, 3, 4 and 5. Therefore, any natural number has one of the following forms :

6k, 6k + 1, 6k + 2, 6k + 3, 6k + 4, 6k + 5.

it is obvious that the numbers 6k, 6k + 2, 6k + 3, and 6k + 4 are composite. Therefore, the prime number $p \geq 5$ has the form 6k + 1 or 6k + 5.

If p = 6k + 1, then $p^2 = (6k + 1)^2 = 36k^2 + 12k + 1$.
If p = 6k + 5, then $p^2 = (6k + 5)^2 = 36k^2 + 60k + 25$
$= 12(3k^2 + 5k + 2) + 1$.

Thus, in both cases, when dividing p^2 by 12, the remainder is equal to 1.

EXAMPLE 15

Prove that for every positive integer n, $1^n + 8^n - 3^n - 6^n$ is divisible by 10.

SOLUTION

Since 10 is the product of two primes 2 and 5, it will suffice to show that the given expression is divisible both by 2 and 5. To do so, we shall use the simple fact that if a and b be any positive integers, then $a^n - b^n$ is always divisible by a –b.
Writing A º $1^n + 8^n - 3^n - 6^n$, $= (8^n - 3^n) - (6^n - 1^n)$,

we find that $8^n - 3^n$ and $6^n - 1^n$ are both divisible by 5, and consequently A is divisible by 5 (= 8 – 3 = 6 – 1). Again, writing $A = (8^n - 6^n) - (3^n - 1^n)$, we find that A is divisible by 2(= 8 – 6 = 3 – 1). Hence A is divisible by 10.

LCM AND HCF

1. HCF is highest common factor between any two or more numbers (or algebraic expression) when only take numbers Its called highest common divisor.
2. LCM is least common multiple between any two or more numbers (or algebraic expression)
3. Multiplication of LCM and HCF of two numbers is equal to multiplication of two numbers.
4. LCM of $\left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{LCM of } (a, p, \ell)}{\text{HCF of } (b, q, m)}$
5. HCF of $\left(\frac{a}{b}, \frac{p}{q}, \frac{\ell}{m}\right) = \frac{\text{HCF of } (a, p, \ell)}{\text{LCM of } (b, q, m)}$
6. LCM of rational and irrational number is not defined.

Remainder Theorem

Let P(x) be any polynomial of degree greater than or equal to one and 'a' be any real number. If P(x)is divided (x – a), then the remainder is equal to P(a).

Factor Theorem

Let P(x) be polynomial of degree greater than of equal to 1 and 'a' be a real number such that P(a) = 0, then (x – a) is a factor of P(x). Conversely, if (x – a) is a factor of P(x), then P(a) = 0.

Some Important Identities

1. $(a + b)^2 = a^2 + 2ab + b^2 = (a - b)^2 + 4ab$
2. $(a - b)^2 = a^2 - 2ab + b^2 = (a + b)^2 - 4ab$
3. $a^2 - b^2 = (a + b)(a - b)$
4. $(a + b)^3 = a^3 + b^3 + 3ab(a + b)$
5. $(a - b)^3 = a^3 - b^3 - 3ab(a - b)$
6. $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = (a + b)(a^2 + b^2 - ab)$
7. $a^3 - b^3 = (a - b)^3 + 3ab(a - b) = (a - b)(a^2 + b^2 + ab)$
8. $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

$= a^2 + b^2 + c^2 + 2abc\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

9. $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

10. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = \frac{1}{2}(a + b + c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$

If $a + b + c = 0$ then $a^3 + b^3 + c^3 = 3abc$

11. $a^4 - b^4 = (a + b)(a - b)(a^2 + b^2)$

12. $a^4 + a^2 + 1 = (a^2 + 1)^2 - a^2 = (1 + a + a^2)(1 - a + a^2)$

Remarks

1. $ab + bc + ca = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

2. $a^2 + b^2 + c^2 - ab - bc - ca = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

Definition Of Indices

If 'a' any none zero real or imaginary number and m is positive integer than a^m = a.a.a. ....a (m times) where 'a' is base 'm' is indices

Law of Indices

1. $a^0 = 1$, $(a \neq 0)$

2. $a^{-m} = \frac{1}{a^m}$, $(a \neq 0)$

3. $a^{m+n} = a^m . a^n$, where m and n real numbers

4. $a^{m-n} = \frac{a^m}{a^n}$, where m and n real numbers, $a \neq 0$

5. $(a^m)^n = a^{mn}$

6. $a^{p/q} = \sqrt[q]{a^p}$

SOLVED EXAMPLE

EXAMPLE 16

Find p and q so that (x + 2) and (x – 1) may be factors of the polynomial $f(x) = x^3 + 10x^2 + px + q$.

SOLUTION

Since (x + 2) is a factor f(–2) must be zero

$\therefore \quad -8 + 40 - 2p + q = 0$ **...(1)**

Since (x – 1) is a factor, f(1) must be zero

$\therefore \quad 1 + 10 + p + q = 0$ **...(2)**

From (1) and (2), by solving we get p = 7 and q = –18

EXAMPLE 17

Show that (2x + 1) is a factor of the expression $f(x) = 32x^5 - 16x^4 + 8x^3 + 4x + 5$.

SOLUTION

Since (2x + 1) is to be a factor of f(x), $f\left(-\frac{1}{2}\right)$ should be zero.

$$f\left(-\frac{1}{2}\right) = 32\left(-\frac{1}{2}\right)^5 - 16\left(-\frac{1}{2}\right)^4 + 8\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right) + 5 .$$

Hence (2x + 1) is a factor of f(x).

EXAMPLE 18

Without using the Remainder theorem, find the remainder when $f(x) = x^6 - 19x^5 + 69x^4 - 151x^3 + 229x^2 + 166x + 26$ is divided by x – 15.

SOLUTION

f(x) can be written as

$(x^6 - 15x^5) - 4(x^5 - 15x^4) + 9(x^4 - 15x^3) - 16(x^3 - 15x^2) - 11(x^2 - 15x) + (x - 15) + 41$

or as $f(x) = x^5(x - 15) - 4x^4(x - 15) + 9x^3(x - 15) - 16x^2(x - 15) - 11x(x - 15) + (x - 15) + 41$

Since the first six terms have x – 15 as a factor, remainder = 41.

EXAMPLE 19

Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

SOLUTION

Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials.

Then $g(x) = x^2 - 3x + 2 = (x - 1)(x - 2)$

In order to prove that f(x) is exactly divisible by g(x), it is sufficient to prove that x – 1 and x – 2 are factors of f(x). For this it is sufficient to prove that f(1) = 0 and f(2) = 0.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow \quad f(1) = 2 \times 1^4 - 6 \times 1^3 + 3 \times 1^2 + 3 \times 1 - 2$ and, $f(2) = 2 \times 2^2 - 6 \times 2^3 \times 2^2 + 3 \times 2 - 2$

$\Rightarrow \quad f(1) = 2 - 6 + 3 + 3 - 2$ and $f(2) = 32 - 48 + 12 + 6 - 2$

$\Rightarrow \quad f(1) = 8 - 8$ and $f(2) = 50 - 50$

$\Rightarrow \quad f(1) = 0$ and $f(2) = 0$

$\Rightarrow \quad (x - 1)$ and $(x - 2)$ are factors of f(x)

$\Rightarrow \quad g(x) = (x - 1)(x - 2)$ is a factors of f(x).

Hence, f(x) is exactly divisible by g(x).

EXAMPLE 20

Using factor theorem, show that a – b, b – c and c – a are the factors of

$$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2).$$

SOLUTION

By factor theorem, a – b will be a factor of the given expression if it vanishes by substituting a = b in it.substituting a = b in the given expression,

we have $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$

$= b(b^2 - c^2) + b(c^2 - b^2) + c(b^2 - b^2)$

$= b^3 - bc^2 + bc^2 - b^3 + c(b^2 - b^2) = 0$

$\therefore$ (a – b) is a factor of $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

Similarly, we can show that (b – c) and (c – a) are also factors of the given expression.

Hence, (a – b), (b – c) and (c – a) are factors of the given expression.

EXAMPLE 21

Show that x – 2y is a factor or $3x^3 - 2x^2y - 13xy^2 + 10y^3$.

SOLUTION

Let $f(x) = 3x^3 - 2x^2y - 13xy^2 + 10y^3$

Then $f(2y) = 3(2y)^3 - 2y(2y)^2 - 13y^2(2y) + 10y^3$

$= 24y^3 - 8y^3 - 26y^3 + 10y^3 = 0$

Hence x – 2y is a factor of f(x).

EXAMPLE 22

Show that $a^n - b^n$ is divisible by a – b if n is any positive integer odd or even.

SOLUTION

Let $a^n - b^n = f(a)$. By Remainder theorem, $f(b) = b^n - b^n = 0$ (replacing a by b)

$\therefore$ a – b is a factor of $a^n - b^n$.

EXAMPLE 23

Show that $a^n - b^n$ is divisible by (a + b) when n is an even positive integer. but not if n is odd.

SOLUTION

Let $a^n - b^n = f(a)$.

Now $f(-b) = (-b)^n - b^n = b^n - b^n = 0$

if n is even and hence a + b is a factor of $a^n - b^n$

If n is odd, $f(-b) = -b^n - b^n = -2b^n \neq 0$.

EXAMPLE 24

If a + b + c = 0,

prove that

$a^4 + b^4 + c^4 = 2(b^2c^2 + c^2a^2 + a^2b^2) = 1/2(a^2 + b^2 + c^2)^2$

SOLUTION

Squaring both sides of the relation

$(a^2 + b^2 + c^2)^2 = [-2(bc + ca + ab)]^2$

$= 4\{b^2c^2 + c^2a^2 + a^2b^2 + 2\{bc.\,ca + ca.\,ab + ab.\,bc\},$

$= 4(b^2c^2 + c^2a^2 + a^2b^2) + 8abc(a + b + c)$

$= 4(b^2c^2 + c^2a^2 + a^2b^2)$, since a + b + c = 0.

Therefore, $2(b^2c^2 + c^2a^2 + a^2b^2) = 1/2\,(a^2 + b^2 + c^2)^2$.

Also $(a^2 + b^2 + c^2)^2 = (a^4 + b^4 + c^4) + 2(b^2c^2 + c^2a^2 + a^2b^2)$,

so that $4(b^2c^2 + c^2a^2 + a^2b^2) = (a^4 + b^4 + c^4) + 2(b^2c^2 + c^2a^2 + a^2b^2)$

where $a^4 + b^4 + c^4 = 2(b^2c^2 + c^2a^2 + a^2b^2)$.

EXAMPLE 25

Solve the equation, $\frac{x - ab}{a + b} + \frac{x - bc}{b + c} + \frac{x - ca}{1 + a} = a + b + c.$

What happens if $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 0$

SOLUTION

$$\left(\frac{x - ab}{a+b} - c\right) + \left(\frac{x - bc}{b+c} - a\right) + \left(\frac{x - ca}{c+a} - b\right) = 0$$

$\Rightarrow$ $(x - (ab + bc + ca))\left[\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right] = 0$

$\Rightarrow$ x = ab + bc + ca.

If $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = 0$

$\Rightarrow$ the given equation becomes an identity & is true for all $x \in R$

RATIO

1. If A and B be two quantities of the same kind, then their ratio is A : B; which may be denoted by the fraction $\frac{A}{B}$ (This may be an integer or fraction)
2. A ratio may represented in a number of ways e.g. $\frac{a}{b} = \frac{ma}{mb} = \frac{na}{nb} =$....where m,n,......are non-zero numbers.
3. To compare two or more ratio, reduce them to common denominator.
4. Ratio between two ratios may be represented as the ratio of two integers e.g. $\frac{a}{b} : \frac{c}{d} : \frac{a/b}{c/d} = \frac{ad}{bc}$ or ad : bc.
5. Ratios are compounded by multiplying them together i.e. $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f}$ ......$= \frac{ace}{bdf}$ ....
6. If a : b is any ratio then its duplicate ratio is $a^2 : b^2$; triplicate ratio is $a^3 : b^3$..... etc.
7. If a : b is any ratio, then its sub-duplicate ratio is $a^{1/2} : b^{1/2}$; sub-triplicate ratio is $a^{1/3} : b^{1/3}$ etc.

PROPORTION

When two ratios are equal, then the four quantities compositing them are said to be proportional.

If $\frac{a}{b}=\frac{c}{d}$, then it is written as a : b = c : d or a : b : : c : d

1. 'a' and 'd' are known as extremes and 'b and c' are known as means.
2. An important property of proportion Product of extremes = product of means.
3. If a : b = c : d, then b : a = d : c (Invertando)
4. If a : b = c : d, then a : c = b : d (Alternando)
5. If a : b = c : d, then $\frac{a+b}{b}=\frac{c+d}{d}$ (Componendo)
6. If a : b = c : d, then $\frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo)
7. If a : b = c : d, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)
8. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=.....$, then each $\frac{a+c+e+.....}{b+d+f+......}$ $=\frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}$
9. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\,..$, then each $=\frac{xa+yc+ze+..}{xb+yd+zf+..}$
10. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\,..$, then each $=\left(\frac{xa^n+yc^n+ze^n}{xb^n+yd^n+zf^n}\right)^{1/n}$

SOLVED EXAMPLE

EXAMPLE 26

If $\frac{x+y}{2}=\frac{y+z}{3}=\frac{z+x}{4}$, then find x : y : z.

SOLUTION

Each $=\frac{\text{Sum of the numerators}}{\text{Sum of the denominators}}=\frac{2(x+y+z)}{9}=\frac{x+y+z}{9/2}$

and therefore each =

$$\frac{(x+y+z)-(y+z)}{\frac{9}{2}-3}=\frac{(x+y+z)-(x+z)}{\frac{9}{2}-4}=\frac{(x+y+z)-(x+y)}{\frac{9}{2}-2}$$

$$=\frac{x}{3/2}=\frac{y}{1/2}=\frac{z}{5/2}\Rightarrow x:y:z=3:1:5$$

EXAMPLE 27

If a(y + z) = b(z + x) = c(x + y),

then show that $\frac{a-b}{x^2-y^2}=\frac{b-c}{y^2-z^2}=\frac{c-a}{z^2-x^2}$

SOLUTION

Given condition can be written as

$$\frac{y+z}{1/a}=\frac{z+x}{1/b}=\frac{x+y}{1/c}=k \quad ....(1)$$

Each

$$=\frac{(z+x)-(y+z)}{\frac{1}{b}-\frac{1}{a}}=\frac{(x+y)-(x+z)}{\frac{1}{c}-\frac{1}{b}}=\frac{(y+z)-(x+y)}{\frac{1}{a}-\frac{1}{c}}$$

$$=\frac{x-y}{\frac{a-b}{ab}}=\frac{y-z}{\frac{b-c}{bc}}=\frac{z-x}{\frac{c-a}{ca}}=k \quad ....(2)$$

Form (1) and (2), we get by multiplying

$$\frac{x^2-y^2}{a-b}=\frac{y^2-z^2}{b-c}=\frac{z^2-x^2}{c-a}$$

$$\Rightarrow \frac{a-b}{x^2-y^2}=\frac{b-c}{y^2-z^2}=\frac{c-a}{z^2-x^2}$$

EXAMPLE 28

If $x=\frac{\sqrt{2a+3b}+\sqrt{2a-3b}}{\sqrt{2a+3b}-\sqrt{2a-3b}}$, show that $3bx^2 - 4ax + 3b = 0$.

SOLUTION

Taking the left hand side as $\frac{x}{1}$, using componendo and dividendo, $\frac{x+1}{x-1}=\frac{\sqrt{2a+3b}}{\sqrt{2a-3b}}$

Squaring, $\frac{(x+1)^2}{(x-1)^2}=\frac{2a+3b}{2a-3b}$ and again applying componendo and dividendo $\frac{x^2+1}{2x}=\frac{2a}{3b}$ which gives the answer on cross multiplication.

EXAMPLE 29

If $\frac{2y+2z-x}{a}=\frac{2z+2x-y}{b}=\frac{2x+2y-z}{c}$, then show that $\frac{9x}{2b+2c-a}=\frac{9y}{2c+2a-b}=\frac{9z}{2a+2b-c}$

SOLUTION

Since $\frac{2y+2z-x}{a}=\frac{2z+2x-y}{b}=\frac{2x+2y-z}{c}$,

each is equal to

$\frac{2(2z+2x-y)+2(2x+2y-z)-(2y+2z-x)}{2b+2c-a}$ by a

theorem quoted earlier $=\frac{9x}{2b+2c-a}$ on simplification.

Similarly, each $=\frac{9y}{2c+2a-b}$ and $\frac{9z}{2a+2b-c}$ and hence the result.

EXAMPLE 30

Solve : $\frac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}=2$

SOLUTION

Writing the R.H.S. as $\frac{2}{1}$ and using componendo and dividendo,

$$\frac{(\sqrt{2+x}+\sqrt{2-x})+(\sqrt{2+x}-\sqrt{2-x})}{(\sqrt{2+x}+\sqrt{2-x})-(\sqrt{2+x}-\sqrt{2-x})}=\frac{2+1}{2-1}$$

(i.e.) $\frac{\sqrt{2+x}}{\sqrt{2-x}}=\frac{3}{1}$

mkSquaring, $\frac{2+x}{2-x}=\frac{9}{1}$ and again applying componendo and dividendo $\frac{4}{2x}=\frac{10}{8}$ and hence $x=\frac{8}{5}$

INTERVALS

Intervals are subsets of R and generally its used to find domain or inequality. If a and b are two real numbers such that

a < b then we can defined for types of intervals

Open Interval (a, b) $\{x : a < x < b\}$

i.e. extreme points are not includes

Closed Interval [a, b] $\{x : a \le x \le b\}$

i.e. extreme points are includes

It can possible when a and b are finite

Semi-Open Interval (a, b] $\{x : a < x \le b\}$

i.e. a is not include and b is include

Semi-Closed Interval [a, b) $\{x : a \le x < b\}$

i.e. a is include and b is not include

Method of Intervals

Let $F(x) = (x-a_1)^{k_1}(x-a_2)^{k_2} .... (x-a_{n-1})^{k_{n-1}}(x-a_n)^{k_n}$.

Here $k_1, k_2 ...., k_n \in Z$ are $a_1, a_2, ...a_n$ ae fixed real numbers satisfying the condition

$a_1 < a_2 < a_3 < ... < a_{n-1} < a_n$

For solving F(x) > 0 or F(x) < 0, consider the following algorithm :

1. We mark the numbers $a_1, a_2, ...a_n$ on the number axis and put plus sign in the interval on the right of the largest of these numbers,i.e. on the right of a_n.
2. Then we put sign in the interval on the left of a_n if k_n is an even number and minus sign if k_n is an odd number. In the next interval, we put a sign according to the following rule :
3. When passing through the point a_{n-1},the polyno mial F(x) changes sign if k_{n-1} is an odd number. Then we consider the next interval and put a sign in it using the same rule.
4. Thus, we consider all the intervals. The solution of the inequality F(x) > 0 is the union of all intervals in which we put plus sign and the solution of the inequality F(x) < 0 is the union of all intervals in which we put minus sign.

Frequently Used Inequalities

1. $(x-a)(x-b) < 0 \Rightarrow x \in (a,b)$. where a < b

2. $(x-a)(x-b) > 0 \Rightarrow x \in (-\infty,a)\cup(b,\infty)$, where a < b

3. $x^2 \le a^2 \Rightarrow x \in [-a,a]$

4. $x^2 \ge a^2 \Rightarrow x \in (-\infty,-a]\cup[a,\infty)$

5. $ax^2+bx+c<0, (a>0) \Rightarrow x \in (\alpha,\beta)$, where α,β $(\alpha<\beta)$ are the roots of the equation $ax^2+bx+c=0$

6. $ax^2+bx+c>0, (a>0)$

$\Rightarrow x \in (-\infty,\alpha)\cup(\beta,\infty)$, where $\alpha,\beta,(\alpha<\beta)$ are the roots of the equation $ax^2+bx+c=0$

SECTION - B : LOG & PROPERTIES

LOGARITHM OF A NUMBER

The logarithm of the number N to the base 'a' is the exponent indicating the power to which the base 'a' must be raised to obtain the number N.

This number is designated as $\log_a N$.

Hence $\ell og_a N = x \Leftrightarrow a^x = N, a > 0, a \ne 1$ & $N > 0$

Common and natural logarithm

$\ell og_{10}N$ is referred as a common logarithm and $\ell og_e N$ is called as natural logarithm of N to the base Napierian and is popularly written as ℓn N. Note that e is an irrational quantity lying between 2.7 to 2.8 **Note that** $e^{\ell n\, x} = x$.

The existence and uniqueness of the number $\ell og_a N$ follows from the properties of an exponential functions.

From the definition of the logarithm of the number N to the base 'a', we have an identity :

$$a^{\log_a N} = N, a > 0, a \ne 1 \;\&\; N > 0$$

This is known as the **FUNDAMENTAL LOGARITHMIC IDENTITY.**

$\ell og_a 1 = 0$ $\quad$ $(a > 0, a \neq 1)$
$\ell og_a a = 1$ $\quad$ $(a > 0, a \neq 1)$
$\ell og_{1/a} a = -1$ $\quad$ $(a > 0, a \neq 1)$

Remember

$\ell og_{10} 2 = 0.3010,$
$\ell og_{10} 3 = 0.4771, \quad \ell n\, 2 = 0.693, \quad \ell n\, 10 = 2.303$

The principal properties of logarithms :

Let M & N are arbitrary positive numbers, $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$ and α is any real number then ;

1. $\ell og_a(M.N) = \ell og_a M + \ell og_a N$
2. $\ell og_a(M/N) = \ell og_a M - \ell og_a N$
3. $\ell og_a M^{\alpha} = \alpha.\ \ell og_a M$
4. $\ell og_{a^{\beta}} M = \dfrac{1}{\beta}\ \ell og_a M$
5. $\ell og_b M = \dfrac{\ell og_a M}{\ell og_a b}$ (base change theorem)

Remarks

1. $\ell og_b a . \ell og_a b = 1 \Leftrightarrow \ell og_b a = \dfrac{1}{\ell og_a b}$
2. $\ell og_b a.\ \ell og_c b.\ \ell og_a c = 1$
3. $\ell og_y x.\ \ell og_z y.\ \ell og_a z = \ell og_a x. \qquad e^{\ell n\, a^x} = a^x$

SOLVED EXAMPLE

EXAMPLE 31

Compute $\sqrt{\left(\dfrac{1}{\sqrt{27}}\right)^{2-\frac{\log_5 13}{2\log_5 9}}}$

SOLUTION

Using in succession the laws of logarithms and exponents we compute the radicand:

$$\left(\frac{1}{\sqrt{27}}\right)^{2-\frac{\log_5 13}{2\log_5 9}} = \frac{1}{27}\cdot(\sqrt{27})^{\frac{1}{2}\log_9 13}$$

$$= \frac{1}{27}\cdot(3^{\log_3 13})^{3/8} = 3^{-3}.\ 13^{3/8}$$

where it is clear that the given number is equal to $3^{-3/2}.\ 13^{3/16}$.

EXAMPLE 32

Compute $\log_{ab}\dfrac{\sqrt[3]{a}}{\sqrt{b}}$ if $\log_{ab} a = 4$.

SOLUTION

By the laws of logarithms we have

$$\log_{ab}\frac{\sqrt[3]{a}}{\sqrt{b}} = \frac{1}{3}\log_{ab} a - \frac{1}{2}\log_{ab} b = \frac{4}{3} - \frac{1}{2}\log_{ab} b$$

It remains to find the quantity $\log_{ab} b$.

Since $\quad 1 = \log_{ab} ab = \log_{ab} a + \log_{ab} b = 4 + \log_{ab} b$

It follows that $\log_{ab} b = -3$ and so

$$\log_{ab}\frac{\sqrt[3]{a}}{\sqrt{b}} = \frac{4}{3} - \frac{1}{2}.(-3) = \frac{17}{6}$$

EXAMPLE 33

Compute the value of $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$.

SOLUTION

$$\frac{1}{\log_2 36} + \frac{1}{\log_3 36} = \log_{36} 2 + \log_{36} 3 = \log_{36} 6 = \frac{1}{2}$$

EXAMPLE 34

If $\log_{x-3}(2x-3)$ is a meaningful quantity then find the interval in which x must lie.

SOLUTION

$x - 3 > 0,\ x - 3 \neq 1$ and $2x - 3 > 0 \Rightarrow x > 3,\ x \neq 4$ and $x > 3/2 \Rightarrow (3, 4) \cup (4, \infty)$

EXAMPLE 35

Given $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8) = \dfrac{2}{s^3+1}$. Write $\log_2\dfrac{a^2 b^5}{c^4}$ as a function of 's' $(a, b, c > 0,\ c \neq 1)$.

SOLUTION

Given $\log_2 a = s$ ....(1)
$\log_2 b = 2s^2$ ....(2)
$\log_8 c^2 = \dfrac{s^3+1}{2}$ ....(3)

$\Rightarrow \quad \dfrac{2\log c}{3\log 2} = \dfrac{s^3+1}{2}$

$\Rightarrow \quad 4\log_2 c = 3(s^3 + 1)$ ....(4)

to find $\ 2\log_2 a + 5\log_2 b - 4\log_2 c$

$\Rightarrow \quad 2s + 10s^2 - 3(s^3 + 1)$

EXAMPLE 36

If $\log 25 = a$ and $\log 225 = b$, then find the value of $\log\left(\left(\dfrac{1}{9}\right)^2\right) + \log\left(\dfrac{1}{2250}\right)$ in terms of a and b (base of the log is 10 everywhere).

SOLUTION

$\log 25 = a; \qquad \log 225 = b$
$2\log 5 = a\ ;\ \log(25 \cdot 9) = b$
or $\log 25 + 2\log 3 = b$

$\Rightarrow \quad 2\log 3 = b - a \quad$ now $\log\left(\frac{1}{9}\right)^2 + \log\left(\frac{1}{2250}\right)$

$= \quad -2\log 9 - \log 2250$

$= \quad -4\log 3 - [\log 225 + \log 10]$

$= -2(b-a) - [b+1]$

$= -2b + 2a - b - 1$

$= 2a - 3b - 1$

EXAMPLE 37

Compute $\log_6 16$ if $\log_{12} 27 = a$

SOLUTION

The chain of transformations

$$\log_6 16 = 4\log_6 2 = \frac{4}{\log_2 6} = \frac{4}{1+\log_2 3}$$

shows us that we have to know $\log_2 3$ in order to find $\log_6 16$. We find it from the condition

$\log_{12} 27 = a$: $a = \log_{12} 27 = 3\log_{12} 3$

$$= \frac{3}{\log_3 12} = \frac{3}{1+2\log_3 2} = \frac{3}{1+\frac{2}{\log_2 3}} = \frac{3\log_2 3}{2+\log_2 3}$$

which means that $\log_2 3 = \frac{2a}{3-a}$

(note that, obviously, $a \neq 3$).

We finally have $\log_6 16 = \frac{4(3-a)}{3+a}$.

SECTION - C : LOG EQUATIONS

LOGARITHMIC EQUATIONS

$\ell og_a x = \ell og_a y$ possible if $x = y$

i.e. $\ell og_a x = \ell og_a y \Leftrightarrow x = y$

Always check the validity of the given equation i.e. $x > 0$, $y > 0$, $a > 0$, $a \neq 1$

SOLVED EXAMPLE

EXAMPLE 38

For $x \geq 0$, what is the smallest possible value of the expression $\log(x^3 - 4x^2 + x + 26) - \log(x + 2)$?

SOLUTION

$$\log\frac{(x^3-4x^2+x+26)}{(x+2)} = \log\frac{(x^2-6x^2+13)(x+2)}{(x+2)}$$

$= \log(x^2 - 6x + 13) \qquad [\because \; x \neq -2]$

$= \log\{(x-3)^2 + 4\}$

$\therefore \quad$ Minimum value is log 4 when $x = 3$

EXAMPLE 39

If $\log_6 15 = \alpha$ and $\log_{12} 18 = \beta$, then compute the value of $\log_{25} 24$ in terms of α & β.

SOLUTION

$$\alpha = \frac{1+\log_3 5}{1+\log_3 2};\ \beta = \frac{2+\log_3 2}{1+2\log_3 2}$$

Let $\log_3 2 = x$ and $\log_3 5 = y$

$1 + y = \alpha(1 + x)$ ....(1)

$2 + x = \beta(2x + 1)$ ....(2)

From (2)

$x = \frac{2-\beta}{2\beta-1}$ ....(3)

Putting this value of x in (1)

$y = \frac{\alpha(1+\beta)-(2\beta-1)}{2\beta-1}$ . ...(4)

Now $\log_{25} 24 = \frac{3x+1}{2y}$. Substitute the value of x and y to

get $\log_{25} 24 = \frac{5-\beta}{2\alpha+2\alpha\beta-4\beta+2}$

EXAMPLE 40

Suppose that *a* and *b* are positive real numbers such that $\log_{27}a + \log_9 b = \frac{7}{2}$ and $\log_{27}b + \log_9 a = \frac{2}{3}$. Find the value of the *ab*.

SOLUTION

$\log_{27}a + \log_9 b = \frac{7}{2} \Rightarrow \frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$;

$\log_{27}b + \log_9 a = \frac{2}{3} \Rightarrow \frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$

adding the equation

$$\frac{1}{3}\log_3(ab) + \frac{1}{2}\log_3(ab) = \frac{7}{2} + \frac{2}{3} = \frac{25}{6}$$

$$\frac{5}{6}\log_3(ab) = \frac{25}{6}$$

$\Rightarrow \quad \log_3(ab) = 5$

$\Rightarrow \quad ab = 3^5 = 243$

EXAMPLE 41

If $\log_2(\log_2(\log_3 x)) = \log_2(\log_3(\log_2 y)) = 0$ then find the value of (x + y).

SOLUTION

$\log_2(\log_2(\log_3 x)) = 0$

$\Rightarrow$ $\log_2(\log_3 x) = 1 \Rightarrow \log_3 x = 2 \Rightarrow x = 9$

$\Rightarrow$ $\log_2(\log_3(\log_2 y)) = 0$

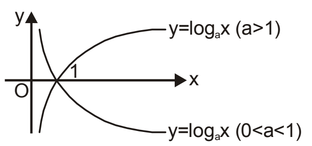
$\Rightarrow$ $\log_3(\log_2 y) = 1 \Rightarrow \log_2 y = 3$

$\Rightarrow$ $y = 8 \therefore x + y = 17$

SECTION - D : LOG INEQUALITIES

STANDARD LOG INEQUALITIES

1. For $a > 1$ the inequality $0 < x < y$ & $\log_a x < \log_a y$ are equivalent.
2. For $0 < a < 1$ the inequality $0 < x < y$ & $\log_a x > \log_a y$ are equivalent.
3. If $a > 1$ then $\log_a x < p \Rightarrow 0 < x < a^p$
4. If $a > 1$ then $\log_a x > p \Rightarrow x > a^p$
5. If $0 < a < 1$ then $\log_a x < p \Rightarrow x > a^p$
6. If $0 < a < 1$ then $\log_a x > p \Rightarrow 0 < x < a^p$



Remarks

1. If the number & the base are on one side of the unity, then the logarithm is positive; If the number and the base are on different sides of unity, then the logarithm is negative.
2. The base of the logarithm 'a' must not equal unity otherwise numbers not equal to unity will not have a logarithm & any number will be the logarithm of unity.
3. For a non negative number 'a' & $n \geq 2$, $n \in N$

   $\sqrt[n]{a} = a^{1/n}$

SOLVED EXAMPLE

EXAMPLE 42

If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, then x lies in the interval

SOLUTION

First we note that for the functions involved in the given inequality to be defined $(x - 1)$ must be greater than 0, that is, $x > 1$.

Now $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow$ $\log_{0.3}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow$ $\log_{0.3}(x-1)^2 < \log_{0.3}(x-1)$

$\Rightarrow$ $(x - 1)^2 > x - 1$

[Note that the inequality is reversed because the base of the logarithms lies between 0 and 1]

$\Rightarrow$ $(x - 1)^2 - (x - 1) > 0$

$\Rightarrow$ $(x - 1)(x - 2) > 0$ …(i)

Since $x > 1$,

therefore the inequality (i) will hold if $x > 2$.

Hence x lies in the interval $(2, \infty)$.

EXAMPLE 43

$x^{\log_5 x} > 5$ then x may belongs to

SOLUTION

$(\log_5 x)^2 > 1$

$\Rightarrow$ $\log_5 x < -1$ or $\log_5 x > 1 \Rightarrow x < \frac{1}{5}$ or $x > 5$

But $x > 0$

$\Rightarrow$ $x \in \left(0, \frac{1}{5}\right) \cup (5, \infty)$

SECTION - E : CHARACTERSTIC & MANTISSA

CHARACTERISTIC & MANTISSA

The common logarithm of a number consists of two parts, integral and fractional, of which the integral part may be zero or an integer (+ve or –ve) and the fractional part a decimal, less than one and always positive.

The integral part is called the *characteristic* and the decimal part is called the *mantissa*. It should be noted that, if the characteristic of the logarithm of N is p then number of significant digit in N = p + 1 if p is the non negative characteristic of log N. Number of zeros after decimal before a significant figure start is p – 1

SOLVED EXAMPLE

EXAMPLE 44

Let $x = (0.15)^{20}$. Find the characteristic and mantissa in the logarithm of x, to the base 10. Assume $\log_{10}2 = 0.301$ and $\log_{10}3 = 0.477$.

SOLUTION

$\log x = \log(0.15)^{20} = 20 \log\left(\frac{15}{100}\right)$

$= 20[\log 15 - 2]$

$= 20[\log 3 + \log 5 - 2]$

$= 20[\log 3 + 1 - \log 2 - 2]$

$= 20[-1 + \log 3 - \log 2]$

$= 20[-1 + 0.477 - 0.301]$

$= -20 \times 0.824 = -16.48 = \overline{17}.52$

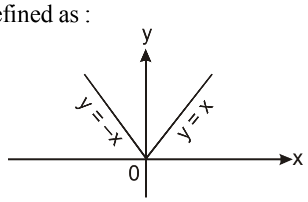
Hence characteristic = – 17 and mantissa = 0.52

SECTION - F
MODULUS EQUATIONS /INEQUALITIES

ABSOLUTE VALUE FUNCTION / MODULUS FUNCTION :

A function $y = |x|$ is called the absolute value function or Modulus function. It is defined as :

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



Remarks

1. $|x| < a \Rightarrow -a < x < a$
2. $|x| > a \Rightarrow x < -a$ or $x > a$

SOLVED EXAMPLE

EXAMPLE 45

Solution of the equation $|x + 1| - |x - 1| = 3$

SOLUTION

$x \geq 1$; $x + 1 - x + 1 = 3$
Not possible

$-1 \leq x < 1$; $x + 1 + x - 1 = 3$
$x = 3/2$ Not possible

$x < -1$; $-x - 1 + x + 1 = 3$
$0 = 3$ Not possible.

EXAMPLE 46

If x satisfies $|x-1|+|x-2|+|x-3| \geq 6$, then

SOLUTION

For $x \leq 1$, the given inequation becomes
$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow -3x \geq 0$

$\Rightarrow$ $x \leq 0$ and for $x \geq 3$, the given equation becomes

$x-1+x-2+x-3 \geq 6 \Rightarrow 3x \geq 12 \Rightarrow x \geq 4$

For $1 < x \leq 2$ we get $x-1+2-x+3-x \geq 6$

$\Rightarrow$ $-x+4 \geq 6$ i.e. $-x \geq 2 \Rightarrow x \leq -2$
not possible.

For $2 < x < 3$,

we get $x-1+x-2+3-x \geq 6$

$\Rightarrow$ $x \geq 6$ not possible.

Hence solution set is $(-\infty, 0] \cup [4, \infty)$
i.e. $x \leq 0$ or $x \geq 4$

Exercise - 1 Objective Problems | JEE Main

SECTION - A : BASIC MATHS

1. If A & B are two rational numbers and AB, A + B and A – B are rational numbers, then A/B is
(A) always rational (B) never rational
(C) rational when $B \neq 0$ (D) rational when $A \neq 0$

2. Every irrational number can be expressed on the number line. This statement is
(A) always true
(B) never true
(C) true subject to some condition
(D) None of these

3. The multiplication of a rational number 'x' and an irrational number 'y' is
(A) always rational
(B) rational except when $y = \pi$
(C) always irrational
(D) irrational except when x = 0

4. If a, b, c are real, then a (a – b) + b (b – c) + c (c – a) = 0, only if
(A) a + b + c = 0
(B) a = b = c
(C) a = b or b = c or c = a
(D) a – b – c = 0

5. If x – a is a factor of $x^3 - a^2x + x + 2$, then 'a' is equal to
(A) 0 (B) 2
(C) – 2 (D) 1

6. If $2x^3 - 5x^2 + x + 2 = (x-2)(ax^2 - bx - 1)$, then a & b are respectively
(A) 2, 1 (B) 2, – 1
(C) 1, 2 (D) –1, 1/2

7. If x, y are rational numbers such that
$(x+y)+(x-2y)\sqrt{2} = 2x - y + (x - y - 1)\sqrt{6}$ then
(A) x = 1, y = 1
(B) x = 2, y = 1
(C) x = 5, y = 1
(D) x & y can take infinitely many values

SECTION - B : LOG PROPERTIES

8. Find the value of the expression
$\frac{2}{\log_4 (2000)^6} + \frac{3}{\log_5 (2000)^6}$.
(A) 6 (B) $\frac{1}{6}$
(C) 5 (D) $\frac{1}{5}$

9. $\frac{1}{1+\log_b a + \log_b c} + \frac{1}{1+\log_c a + \log_c b} + \frac{1}{1+\log_a b + \log_a c}$
has the value equal to
(A) abc (B) $\frac{1}{abc}$
(C) 0 (D) 1

10. Greatest integer less than or equal to the number $\log_2 15 \,.\, \log_{1/6} 2 \,.\, \log_3 1/6$ is
(A) 4 (B) 3
(C) 2 (D) 1

11. Anti logarithm of 0.75 to the base 16 has the value equal to
(A) 4 (B) 6
(C) 8 (D) 12

12. The number $\log_2 7$ is
(A) an integer (B) a rational number
(C) an irrational number (D) a prime number

13. The ratio $\frac{2^{\log_{2^{1/4}} a} - 3^{\log_{27}(a^2+1)^3} - 2a}{7^{4\log_{49} a} - a - 1}$ simplifies to
(A) $a^2 - a - 1$ (B) $a^2 + a - 1$
(C) $a^2 - a + 1$ (D) $a^2 + a + 1$

14. $\frac{1}{\log_{\sqrt{bc}} abc} + \frac{1}{\log_{\sqrt{ca}} abc} + \frac{1}{\log_{\sqrt{ab}} abc}$ has the value equal to
(A) 1/2 (B) 1
(C) 2 (D) 4

SECTION - C : LOG EQUATIONS

15. If $3^{2\log_3 x} - 2x - 3 = 0$, then the number of values of 'x' satisfying the equation is
(A) zero (B) 1
(C) 2 (D) more than 2

16. If $\log_x \log_{18}(\sqrt{2}+\sqrt{8}) = \frac{1}{3}$. Then the value of 1000 x is equal to
(A) 8 (B) 1/8
(C) 1/125 (D) 125

17. Number of real solution (x) of the equation $|x-3|^{3x^2-10x+3} = 1$ is
(A) exactly four (B) exactly three
(C) exactly two (D) exactly one

18. Number of real solution of the equation $\sqrt{\log_{10}(-x)} = \log_{10}\sqrt{x^2}$ is
(A) none (B) exactly 1
(C) exactly 2 (D) 4

19. $\log_4 \log_3 \log_2 x = 0$
(A) 16 (B) 8
(C) 4 (D) None

20. $2 \log_4 (4 - x) = 4 - \log_2 (-2 - x)$.
(A) –4 (B) 4
(C) 3 (D) None

21. $\log_{10}^2 x + \log_{10} x^2 = \log_{10}^2 2 - 1$
(A) $\frac{1}{5}$ (B) $\frac{1}{6}$
(C) $\frac{1}{7}$ (D) None

SECTION - D : LOG INEQUALITIES

22. Solve $\log_2 \frac{x-1}{x-2} > 0$
(A) x > 2 (B) x < 2
(C) $x \le 3$ (D) x > 1

23. Solve $\log_{0.04}(x - 1) \ge \log_{0.2}(x - 1)$
(A) $x \in (1,2]$ (B) $x \le 2$
(C) $x \ge 1$ (D) $x \le 1$

24. Solve $\log_2(x - 1) > 4$
(A) x > 8 (B) x > 17
(C) x > 9 (D) x > 29

25. Solve $\log_{(x+3)}(x^2 - x) < 1$
(A) $x \in (-3,-2) \cup (-1,0) \cup (1,2)$
(B) $x \in (-3,-2) \cup (-1,0) \cup (1,3)$
(C) $x \in (-3,-1) \cup (-1,0) \cup (1,2)$
(D) $x \in (-3,-1) \cup (-1,0) \cup (1,3)$

SECTION - E : CHARACTERSTIC & MANTISSA

26. How many digits are contained in the number 2^{75} ?
(A) 21 (B) 22
(C) 23 (D) 24

27. Let 'm' be the number of digits in 3^{40} and 'p' be the number of zeroes in 3^{-40} after decimal before starting a significant digit the (m + p) is (log3 = 0.4771)
(A) 40 (B) 39
(C) 41 (D) 38

28. Given that log(2) = 0.3010..... the number of digits in the number 2000^{2000} is
(A) 6601 (B) 6602
(C) 6603 (D) 6604

29. If P is the number of integers whose logarithms to the base 10 have the characteristic p , and Q the number of integers the logarithms of whose reciprocals to the base 10 have the characteristic –q , Find value of $\log_{10}P - \log_{10}Q$ is :
(A) p + q – 1 (B) p – q + 1
(C) p + q + 1 (D) None

SECTION - F : MODULUS EQUATIONS / INEQUALITIES

30. The number of real roots of the equation $|x|^2 - 3|x| + 2 = 0$ is
(A) 1 (B) 2
(C) 3 (D) 4

31. Solution of $|4x + 3| + |3x - 4| = 12$ is
(A) $x = -\frac{7}{3}, \frac{3}{7}$ (B) $x = -\frac{5}{2}, \frac{2}{5}$
(C) $x = -\frac{11}{7}, \frac{13}{7}$ (D) $x = -\frac{3}{7}, \frac{7}{5}$

32. $|x - 3| + 2|x + 1| = 4$
(A) –1 (B) 1
(C) 0 (D) None

33. $|x|^2 - |x| + 4 = 2x^2 - 3|x| + 1$
(A) 3 (B) 2
(C) 0 (D) 1

SECTION - G : MIXED PROBLEM

34. If $\log_{10}2 = 0.3010$ & $\log_{10}3 = 0.4771$, find the value of $\log_{10}(2.25)$.
(A) 0.3522 (B) 0.03522
(C) 1.3522 (D) None

35. The value of the expression $\log_{10}(\tan 6°) + \log_{10}(\tan 12°) + \log_{10}(\tan 18°) + \ldots\ldots\ldots + \log_{10}(\tan 84°)$ is
(A) a whole number
(B) an irrational number
(C) a negative integer
(D) a rational number which is not an integer

36. Let ABC be a triangle right angled at C. The value of $\frac{\log_{b+c} a + \log_{c-b} a}{\log_{b+c} a \cdot \log_{c-b} a}$ $(b + c \ne 1, c - b \ne 1)$ equals
(A) 1 (B) 2
(C) 3 (D) 1/2